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Related Expressions for Bernoulli's and Euler's Numbers.

BY J. C. FIELDS.

By the formula*

$$\left(\frac{d}{dx}\right)^n \phi(u) = \sum_1^n \sum_r^n \frac{(-1)^{\rho-r} u^{\rho-r} \phi^{\rho}(u)}{r! (\rho-r)!} \left(\frac{d}{dx}\right)^n u^r, \quad (1)$$

we may very readily connect Bernoulli's and Euler's numbers.

Put $u = e^{-ix}$, then

$$\sec x + \tan x = \frac{2}{u+i} + i = \sum \frac{E_n x^n}{n!}, \quad (2)$$

where, when n is even, E_n is one of Euler's numbers, and when n is odd, Bernoulli's number $B_{\frac{n+1}{2}} = \frac{(n+1) E_n}{2^{n+1}(2^{n+1}-1)}$. Putting $\phi(u) = \frac{1}{u+i}$ in (1), we get

$$\left(\frac{d}{dx}\right)^n \frac{1}{u+i} = \sum_1^n \sum_r^n \frac{(-1)^r \rho! u^{\rho-r} (u+i)^{-(\rho+1)}}{r! (\rho-r)!} \left(\frac{d}{dx}\right)^n e^{-rix}, \quad (3)$$

whence

$$\begin{aligned} \frac{1}{2} E_n &= \left(\frac{d}{dx}\right)^n \frac{1}{u+i} = \sum_1^n \sum_r^n \frac{(-1)^r \rho! (-ri)^n (1+i)^{-(\rho+1)}}{r! (\rho-r)!} \\ &= \sum_1^n \sum_r^n \frac{(-1)^r (-ri)^n}{r!} y^{r+1} \left(\frac{d}{dy}\right)^r y^{\rho} = \sum_1^n \frac{(-1)^r (-ri)^n}{r!} y^{r+1} \left(\frac{d}{dy}\right)^r \frac{1-y^{n+1}}{1-y} \\ &= \sum_1^n \frac{(-1)^r (-ri)^n}{r!} y^{r+1} \left\{ \frac{r!}{(1-y)^{r+1}} = \sum_0^r r! \binom{n+1}{s} \frac{y^{n-s+1}}{(1-y)^{r-s+1}} \right\} \\ &= \sum_1^n (-1)^r (-ri)^n \left(\frac{y}{1-y}\right)^{r+1} \left\{ 1 - y^{n+1} \sum_0^r \binom{n+1}{s} \left(\frac{1-y}{y}\right)^s \right\} \end{aligned}$$

*For demonstration of this formula see Bertrand's *Calcul Différentiel*, p. 141, or *Amer. Journ. of Math.*, Vol. XI, p. 390, formula (7).

$$\begin{aligned}
&= y^{n+1} \sum_1^n (-1)^r (-ri)^n \left(\frac{y}{1-y}\right)^{r+1} \sum_{r+1}^{n+1} \binom{n+1}{s} \left(\frac{1-y}{y}\right)^s \\
&= y^{n+1} \sum_2^{n+1} \sum_1^{s-1} \binom{n+1}{s} (-1)^r (-ri)^n \left(\frac{1-y}{y}\right)^{s-r-1} \\
&= \left(-\frac{1+i}{2}\right)^{n+1} \sum_2^{n+1} \sum_1^{s-1} i^s \binom{n+1}{s} \cdot i^r r^n.
\end{aligned}$$

Observing that $(1+i)^2 = 2i$, we find

$$\left. \begin{aligned} E_n &= 2 \left(\frac{i}{2}\right)^{\frac{n+1}{2}} \sum_2^{n+1} \sum_1^{s-1} i^{s+r} \binom{n+1}{s} r^n, & (n \text{ odd}) \\ E_n &= -(1+i) \left(\frac{i}{2}\right)^{\frac{n}{2}} \sum_2^{n+1} \sum_1^{s-1} i^{s+r} \binom{n+1}{s} r^n, & (n \text{ even}). \end{aligned} \right\} \quad (4)$$

When n is odd we have then

$$B_{\frac{n+1}{2}} = \frac{2(n+1)}{2^{n+1}(2^{n+1}-1)} \left(\frac{i}{2}\right)^{\frac{n+1}{2}} \sum_2^{n+1} \sum_1^{s-1} i^{s+r} \binom{n+1}{s} r^n.$$

Taking account only of the real terms in the summations on the right side of (4), since E_n being real, the imaginary terms must cancel one another, we may also write (4) under the form

$$\left. \begin{aligned} E_n &= (-1)^{\left[\frac{n}{4}\right] + (-1)^{\frac{n+1}{2}} 2^{-\frac{n+1}{2}}} \\ &\quad \sum_2^{n+1} \binom{n+1}{s} \left\{ \left(1 - (-1)^{\frac{n+1}{2}}\right) v_{s-1} - \left(1 + (-1)^{\frac{n+1}{2}}\right) v_{s-2} \right\}, & (n \text{ odd}) \\ E_n &= (-1)^{\left[\frac{n}{4}\right]} 2^{-\frac{n}{2}} \sum_2^{n+1} \binom{n+1}{s} \left\{ v_{s-1} + (-1)^{\frac{n}{2}} v_{s-2} \right\}, & (n \text{ even}) \end{aligned} \right\} \quad (5)$$

where $(-1)^s v_s = s^n - (s-2)^n + (s-4)^n - \dots$